

Research Article

Generalized Caristi's Fixed Point Theorems

Abdul Latif

Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Abdul Latif, latifmath@yahoo.com

Received 27 December 2008; Accepted 9 February 2009

Recommended by Mohamed A. Khamsi

We present generalized versions of Caristi's fixed point theorem for multivalued maps. Our results either improve or generalize the corresponding generalized Caristi's fixed point theorems due to Bae (2003), Suzuki (2005), Khamsi (2008), and others.

Copyright © 2009 Abdul Latif. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

A number of extensions of the Banach contraction principle have appeared in literature. One of its most important extensions is known as Caristi's fixed point theorem. It is well known that Caristi's fixed point theorem is equivalent to Ekeland variational principle [1], which is nowadays an important tool in nonlinear analysis. Many authors have studied and generalized Caristi's fixed point theorem to various directions. For example, see [2–8]. Kada et al. [9] and Suzuki [10] introduced the concepts of w -distance and τ -distance on metric spaces, respectively. Using these generalized distances, they improved Caristi's fixed point theorem and Ekeland variational principle for single-valued maps. In this paper, using the concepts of w -distance and τ -distance, we present some generalizations of the Caristi's fixed point theorem for multivalued maps. Our results either improve or generalize the corresponding results due to Bae [4, 11], Kada et al. [9], Suzuki [8, 10], Khamsi [5], and many of others.

Let X be a metric space with metric d . We use 2^X to denote the collection of all nonempty subsets of X . A point $x \in X$ is called a fixed point of a map $f : X \rightarrow 2^X$ ($T : X \rightarrow 2^X$) if $x = f(x)$ ($x \in T(x)$).

In 1976, Caristi [12] obtained the following fixed point theorem on complete metric spaces, known as Caristi's fixed point theorem.

Theorem 1.1. *Let X be a complete metric space with metric d . Let $\psi : X \rightarrow [0, \infty)$ be a lower semicontinuous function, and let $f : X \rightarrow X$ be a single-valued map such that for any*

$x \in X$,

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)). \quad (1.1)$$

Then f has a fixed point.

To generalize Theorem 1.1, one may consider the weakening of one or more of the following hypotheses: (i) the metric d ; (ii) the lower semicontinuity of the real-valued function φ ; (iii) the inequality (1.1); (iv) the function f .

In [9], Kada et al. introduced a concept of w -distance on a metric space as follows.

A function $\omega : X \times X \rightarrow [0, \infty)$ is a w -distance on X if it satisfies the following conditions for any $x, y, z \in X$:

- (w₁) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$;
- (w₂) the map $\omega(x, \cdot) : X \rightarrow [0, \infty)$ is lower semicontinuous;
- (w₃) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Clearly, the metric d is a w -distance on X . Let $(Y, \|\cdot\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \rightarrow [0, \infty)$ defined by $\omega_1(x, y) = \|y\|$ and $\omega_2(x, y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w -distances. Many other examples of w -distance are given in [9, 13]. Note that, in general, for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$, and neither of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily holds.

In the sequel, otherwise specified, we shall assume that X is a complete metric space with metric d , $\varphi : X \rightarrow [0, \infty)$ is a lower semicontinuous function and ω is a w -distance on X .

Using the concept of w -distance, Kada et al. [9] generalized Caristi's fixed point theorem as follows.

Theorem 1.2. *Let f be a single-valued self map on X such that for every $x \in X$,*

$$\varphi(f(x)) + \omega(x, f(x)) \leq \varphi(x). \quad (1.2)$$

Then, there exists $x_0 \in X$ such that $f(x_0) = x_0$ and $\omega(x_0, x_0) = 0$.

2. The Results

Applying Theorem 1.2, first we prove the following generalization of Theorem 1.1.

Theorem 2.1. *Let $g : X \rightarrow (0, \infty)$ be any function such that for some $r > 0$,*

$$\sup \left\{ g(x) : x \in X, \varphi(x) \leq \inf_{z \in X} \varphi(z) + r \right\} < \infty. \quad (2.1)$$

Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$\omega(x, y) \leq g(x)(\varphi(x) - \varphi(y)). \quad (2.2)$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Define a function $f : X \rightarrow X$ by $f(x) = y \in T(x) \subseteq X$. Note that for each $x \in X$, we have

$$\omega(x, f(x)) \leq g(x)(\psi(x) - \psi(f(x))). \quad (2.3)$$

Now, since $g(x) > 0$, it follows that $\psi(f(x)) \leq \psi(x)$. Put

$$M = \left\{ x \in X : \psi(x) \leq \inf_{z \in X} \psi(z) + r \right\}, \quad \alpha = \sup_{z \in M} g(z) < \infty. \quad (2.4)$$

Note that M is nonempty, and by the lower semicontinuity of ψ and $\omega(x, \cdot)$, M is closed subset of a complete metric space X , and hence it is complete. Now, we show that $f(M) \subseteq M$. Let $u \in M$, and $f(u) = v \in T(u)$, then we have

$$\psi(f(u)) \leq \psi(u) \leq \inf_{z \in X} \psi(z) + r, \quad (2.5)$$

and thus $f(u) \in M$, and hence f is a self map on M . Note that $\alpha\psi$ is lower semicontinuous and for each $x \in M$, we have

$$\omega(x, f(x)) \leq \alpha(\psi(x)) - \alpha(\psi(f(x))). \quad (2.6)$$

By Theorem 1.2, there exists $x_0 \in M$ such that $f(x_0) = x_0 \in T(x_0)$ and $\omega(x_0, x_0) = 0$. \square

Now, applying Theorem 2.1, we obtain generalized Caristi's fixed point results.

Theorem 2.2. *Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq \max\{c(\psi(x)), c(\psi(y))\}(\psi(x) - \psi(y)), \quad (2.7)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is an upper semicontinuous function from the right. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Put $t_0 = \inf_{x \in X} \psi(x)$. By the definition of the function c , there exist some positive real numbers r, r_0 such that $c(t) \leq r_0$ for all $t \in [t_0, t_0 + r]$. Now, for all $x \in X$, we define

$$g(x) = \max\{c(\psi(x)), c(\psi(y))\}. \quad (2.8)$$

Clearly, g maps x into $(0, \infty)$. Note that for all $x \in X$, we get $\psi(y) \leq \psi(x)$, and thus for any $x \in X$ with $\psi(x) \leq t_0 + r$, we have

$$\psi(y) \leq t_0 + r. \quad (2.9)$$

Now, clearly, $g(x) \leq r_0 < \infty$ and hence we obtain

$$\sup \left\{ g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r \right\} < \infty. \quad (2.10)$$

By Theorem 2.1, T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$. \square

Theorem 2.3. *Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq c(\psi(x))(\psi(x) - \psi(y)), \quad (2.11)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. For each $x \in X$, define $g(x) = c(\psi(x))$. Clearly, g does carry x into $(0, \infty)$. Now, since the function c is nondecreasing, for any real number $r > 0$ we have

$$\sup \left\{ g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r \right\} \leq c\left(\inf_{z \in X} \psi(z) + r\right) < \infty. \quad (2.12)$$

Thus, by Theorem 2.1, the result follows. \square

Corollary 2.4. *Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying*

$$\omega(x, y) \leq c(\psi(y))(\psi(x) - \psi(y)), \quad (2.13)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Since for each $x \in X$ there is $y \in T(x)$ such that $\psi(y) \leq \psi(x)$ and the function c is nondecreasing, we have $c(\psi(y)) \leq c(\psi(x))$. Thus the result follows from Theorem 2.3. \square

Applying Theorem 2.3, we prove the following fixed point result.

Theorem 2.5. *Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $\omega(x, y) \leq \eta(x)(\psi(x) - \psi(y))$ and*

$$\omega(x, y) \leq \eta(\omega(x, y))(\psi(x) - \psi(y)), \quad (2.14)$$

where $\eta : [0, \infty) \rightarrow (0, \infty)$ is an upper semicontinuous function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Define a function c from $[0, \infty)$ into $(0, \infty)$ by

$$c(t) = \sup \{ \eta(r) : 0 \leq r \leq t \}. \quad (2.15)$$

Clearly, c is nondecreasing function. Now, since $\omega(x, y) \leq \psi(x)$, we have $c(\omega(x, y)) \leq c(\psi(x))$. Thus by Theorem 2.3, the result follows. \square

The following result can be seen as a generalization of [5, Theorem 4].

Corollary 2.6. *Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a lower semicontinuous function such that*

$$\limsup_{t \rightarrow 0^+} \frac{t}{\phi(t)} < \infty. \quad (2.16)$$

Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $\omega(x, y) \leq \psi(x)$ and

$$\phi(\omega(x, y)) \leq \psi(x) - \psi(y). \quad (2.17)$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Proof. Define a function $\eta : [0, \infty) \rightarrow (0, \infty)$ by

$$\eta(0) = \limsup_{t \rightarrow 0^+} \frac{t}{\phi(t)}, \quad \eta(t) = \frac{t}{\phi(t)}, \quad t > 0. \quad (2.18)$$

Then η is upper semicontinuous. Also note that

$$\omega(x, y) \leq \eta(\omega(x, y))(\psi(x) - \psi(y)). \quad (2.19)$$

Thus by Theorem 2.5, T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$. \square

Now, let p be a τ distance on X [8], using the same technique as in the proof of Theorem 2.1, and applying [8, Theorem 3], we can obtain the following result.

Theorem 2.7. *Let $g : X \rightarrow (0, \infty)$ be any function such that for some $r > 0$,*

$$\sup \left\{ g(x) : x \in X, \psi(x) \leq \inf_{z \in X} \psi(z) + r \right\} < \infty. \quad (2.20)$$

Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq g(x)(\psi(x) - \psi(y)). \quad (2.21)$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Now, following similar methods as in the proofs of Theorems 2.2, 2.3, 2.5, and Corollaries 2.4 and 2.6, we can obtain the following generalizations of Caristi's fixed point theorem with respect to τ -distance.

Theorem 2.8. Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq \max\{c(\varphi(x)), c(\varphi(y))\}(\varphi(x) - \varphi(y)), \quad (2.22)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is an upper semicontinuous from the right. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Theorem 2.9. Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq c(\varphi(x))(\varphi(x) - \varphi(y)), \quad (2.23)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Corollary 2.10. Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq c(\varphi(y))(\varphi(x) - \varphi(y)), \quad (2.24)$$

where $c : [0, \infty) \rightarrow (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Theorem 2.11. Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \leq \varphi(x)$ and

$$p(x, y) \leq \eta(p(x, y))(\varphi(x) - \varphi(y)), \quad (2.25)$$

where $\eta : [0, \infty) \rightarrow (0, \infty)$ is an upper semicontinuous function. Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Corollary 2.12. Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a lower semicontinuous function such that

$$\limsup_{t \rightarrow 0^+} \frac{t}{\phi(t)} < \infty. \quad (2.26)$$

Let $T : X \rightarrow 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \leq \varphi(x)$ and

$$\phi(p(x, y)) \leq \varphi(x) - \varphi(y). \quad (2.27)$$

Then T has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Similar generalizations of Caristi's fixed point theorem in the setting of quasi-metric spaces with respect to ω -distance and with respect to Q -function are studied in [3, Theorem 5.1(iii), Theorem 5.2] and in [2, Theorem 4.1], respectively.

Acknowledgment

The author is thankful to the referees for their valuable comments and suggestions.

References

- [1] I. Ekeland, "Nonconvex minimization problems," *Bulletin of the American Mathematical Society*, vol. 1, no. 3, pp. 443–474, 1979.
- [2] S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, "Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 1, pp. 126–139, 2008.
- [3] Q. H. Ansari, "Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 561–575, 2007.
- [4] J. S. Bae, "Fixed point theorems for weakly contractive multivalued maps," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 690–697, 2003.
- [5] M. A. Khamsi, "Remarks on Caristi's fixed point theorem," *Nonlinear Analysis: Theory, Methods & Applications*. In press.
- [6] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.
- [7] S. Park, "On generalizations of the Ekeland-type variational principles," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 39, no. 7, pp. 881–889, 2000.
- [8] T. Suzuki, "Generalized Caristi's fixed point theorems by Bae and others," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 502–508, 2005.
- [9] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.
- [10] T. Suzuki, "Generalized distance and existence theorems in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.
- [11] J. S. Bae, E. W. Cho, and S. H. Yeom, "A generalization of the Caristi-Kirk fixed point theorem and its applications to mapping theorems," *Journal of the Korean Mathematical Society*, vol. 31, no. 1, pp. 29–48, 1994.
- [12] J. Caristi, "Fixed point theorems for mappings satisfying inwardness conditions," *Transactions of the American Mathematical Society*, vol. 215, pp. 241–251, 1976.
- [13] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Application*, Yokohama Publishers, Yokohama, Japan, 2000.